MATH 2230 B 10/02/2020

1 Upper bounds for Moduli of Contour Integral.

If $w(t)$ is piecewise cts $\mathbb{C}$-valued function on $[a, b]$, then

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt.$$

Proof: $\int_a^b w(t)\, dt = \boxed{r_0} e^{i\theta_0}$ → Modulus of the integral.

$$\Rightarrow \quad r_0 = \int_a^b e^{-i\theta_0} w(t)\, dt$$

$\mathbb{R}$.

$$\Rightarrow \quad r_0 = \mathrm{Re} \int_a^b e^{-i\theta_0} w(t)\, dt$$

$$= \int_a^b \mathrm{Re}\left(e^{-i\theta_0} w(t)\right) dt.$$

$$\le \int_a^b \left| e^{-i\theta_0} w(t) \right| dt$$

$$= \int_a^b |w(t)|\, dt$$

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt. \quad \square$$

If $|f| \le M$ for all the pt on a contour γ.

$$\left|\int_\gamma f \, dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right|$$

$$\le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\overset{|f|\le M}{\le} M \boxed{\int_a^b |\gamma'(t)|\,dt}$$

the Length of γ.

So define the length to be L.

$$\le ML.$$

2. Antiderivative

Def: For a cts function f on a domain D, we have F s.t $F'(z) = f$ for $\forall z \in D$.

Thm: Suppose f is cts in D. The following 3 are equivalent:

i) f has an antiderivative in D.

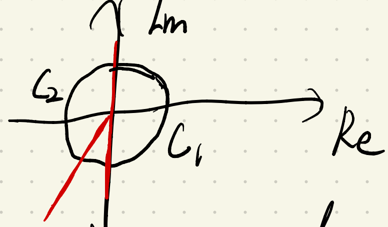
ii) the integral of F along contour lying entirely in D from z_1 to z_2

$$\int_{z_1}^{z_2} f(z)\, dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1).$$

(iii) the integral of f for any closed contour are zero.

Remark: This thm does not tell anything about this cts function, it says only that if any one of the 3 statement is true (false) then the any other two will be true (false).

e.g. $\int_C \frac{1}{z} dz$



We cannot integral using antiderivative directedly.

i) Set $z = re^{i\theta}$ $[\theta \to 2\pi)$, $dz = re^{i\theta} d\theta$

$$\int_0^{2\pi} \frac{1}{re^{i\theta}} ire^{i\theta} d\theta = \int_0^{2\pi} i \, d\theta = 2\pi i$$

$\to$ Residues Thm.

ii) first integrate from $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$F = \text{Log} z$

$$\int_{C_1} \frac{1}{z} dz = \text{Log} z \Big|_{-i}^{i} = \text{Arg}(i) - \text{Arg}(-i)$$
$$= \pi i$$

For C_2, $\frac{\pi}{2} \le \theta \le \frac{3}{2}\pi$,

take $F = \log z$, $0 < \theta < 2\pi$

$$\int_{C_2} \frac{1}{z}\,dz = \log z \Big|_{i}^{-i} = \frac{3}{2}\pi i - \frac{1}{2}\pi i = \pi i$$

$$\int_C \frac{1}{z} = \int_{C_1} + \int_{C_2} \frac{1}{z} = 2\pi i.$$

Some questions from Homework.

P119 - 2

(c) $\int_0^{\pi/6} e^{i2t}\,dt = \frac{1}{2i} e^{i2t}\Big|_0^{\pi/6} = \frac{1}{2i}\left(e^{i\frac{\pi}{3}} - 1\right)$

$$= \frac{1}{2i}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i - 1\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{i}{4}.$$

(d) $\int_0^{\infty} e^{-zt}\,dt = \lim_{a\to\infty} \int_0^a e^{-zt}\,dt$

$$= \lim_{a\to\infty} -\frac{1}{z} e^{-zt}\Big|_0^a$$

$$= \lim\left(-\frac{1}{z} e^{-za} + \frac{1}{z}\right)$$

$$= \lim\left(\frac{1}{z}\left(1 - e^{-za}\right)\right) = \frac{1}{z}$$

$\operatorname{Re} z > 0$ so $|e^{-za}| = e^{-xa} \to 0$, as $a \to \infty$

P119 - 4

Integrate the LHS, then

recall $\begin{cases} \text{Re} \int f = \int \text{Re} f \\ \text{Im} \int f = \int \text{Im} f. \end{cases}$

P124 - 6

a) $x>0$ $x^3 \sin(\pi/x) = 0 \Rightarrow x = \frac{1}{n}$, $n \in \mathbb{Z}$.

$x=0$, by def of $y(x)$, $y=0$,

b) Smooth: The arc $z = z(t)$ $(a \leq t \leq b)$, is C^1 on $[a, b]$ & $z'(t) > 0$ on (a, b)

$z(t) = t + t^3 \sin(\pi/t)\, i$

$z'(t) = 1 + (t^3 \sin(\pi/t))'\, i \neq 0$ on $[0, 1)$.

& as a product of polynomial and a trig function it is C^1 for $[0, 1]$.

$\boxed{\lim_{t \to 0} \frac{z(t) - 0}{t}} = \lim_{t \to 0} (1 + t^2 \sin(\pi/t)\, i)$

Notice $0 \leq |t^2 \sin(\pi/t)\, i| \leq t^2$

By squeeze thm, it converges to 0,

$z'(t) = 1 + (3t^2 \sin(\pi/t) - t' \cos(\pi/t)\pi) i$

$3t^2 \sin(\pi/t) \to 0$ as $t \to 0$,

$0 \leq |t \cos(\pi/t)\pi| \leq t$, so by squeeze thm

again, so $z'(t) \to 1$, as $t \to 0$.

p 132 - 13

$\int_{C_0} (z - z_0)^{n-1} dz$ on $z = z_0 + Re^{i\theta}, -\pi \leq \theta \leq \pi$.

$= \int_{-\pi}^{\pi} R^{n-1} e^{i(n-1)\theta} iRe^{i\theta} d\theta$

$= iR^{n} \int_{-\pi}^{\pi} e^{in\theta} d\theta = \begin{cases} 2\pi i, & n = 0 \\ 0, & n \neq 0 \end{cases}$

$\frac{iR^n}{in} e^{in\theta} \Big|_{-\pi}^{\pi}$

P139 - 4.

$\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \to (1)$, $f = \frac{2z^2 - 1}{z^4 + 5z^2 + 4} = \frac{2z^2 - 1}{(z^2+4)(z^2+1)}$

then $|f| \leq \frac{2|z|^2 + 1}{(|z|^2 - 4)(|z|^2 - 1)} \leq \frac{2R^2 + 1}{(R^2 - 4)(R^2 - 1)}$

now for $|z| = R$, $R > 2$

If C_R is upper half of circle,

$|(1)| \leq \frac{2R^2 + 1}{(R^2 - 4)(R^2 - 1)} \pi R$, if we divide R^4,

$$\frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)} = \frac{\frac{\pi(2+\frac{1}{R^2})}{R}}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})}$$

as $R \to \infty$, $|I(z)| \to 0$.

P147. 5.

Remember to change the branch cut so that the contour contains entirely in the domain.

(Notice that we also need $\log z$ for new branch cut to have same value on the overlapping the old ones.)